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RESEARCH ARTICLE

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Heterogeneous beliefs, the term structure and time-varying risk premia

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Abstract This paper demonstrates theoretically and empirically that one possible economic explanation of the dynamics of the term structure of interest rates is the time-varying heterogeneous beliefs about future economic conditions. Assuming that each agent forms heterogeneous expectations about both his income shock and others' beliefs about their income shocks each period, the paper illustrates that heterogeneous beliefs generate time-varying risk premia of the term structure in a closed-form solution. Motivated by this theory, several empirical tests are conducted using the cross-sectional mean and dispersion of belief indices that are extracted as the differences between non-judgemental econometric forecasts based on diffusion indices in Stock and Watson (J Bus Econ Stat, 2002) and professional survey forecasts. It is shown that (a) an increase in the mean belief about inflation steepens the yield curve, (b) the mean and dispersion of interest rate beliefs help explain the mean and the stochastic volatility of the term structure, suggesting that time-varying risk premia may be explained by endogenous uncertainty caused by heterogeneous beliefs in the economy.

Keywords and phrases Heterogeneous beliefs · The term structure · Survey forecasts

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1 Introduction

The term structure of interest rates is one of the most studied problems in financial economics. However, fundamentals such as consumption, productivity and preferences are unable to explain the dynamics of the term structure well. The consumption-based CAPM developed by Lucas (1978) and Breeden (1979), and later applied to the term structure in Cox, Ingersoll and Ross (1985) and Backus et al. (1989) implies the expectations hypothesis that cannot account for the observed fluctuations in bond returns (e.g see results of Fama and Bliss 1987; Shiller and McCulloch 1990; Campbell and Shiller 1991). Arbitrage-free dynamic affine factor models, as in Litterman and Scheinkman (1991), Pearson and Sun (1994), and Dai and Singleton (2000), do not provide economic interpretations of these latent factors or the formation of time-varying risk premia. As shown in Ang and Piazzesi (2003), the latent yield factors, which depend upon observed yields and have no connection with macroeconomic fundamentals, account for most of the movements of the yield curve. This paper investigates how heterogeneous beliefs, as a result of diverse judgements and modelling based on the same aggregate information, affect the term structure movements and explain the excess volatility in bond returns.

Allen, Morris and Shin (2006) and Bacchetta and Wincoop (2004) adopt asymmetric private information as the source of heterogeneous expectations while Harrison and Kreps (1978), Varian (1985), and Basak (2000) attribute heterogeneous expectations to different subjective probability beliefs. Kurz (1994) and Kurz (1997) initiated the rational heterogeneous beliefs theory which states that rational beliefs are induced by different modelling and diverse interpretations based on common information and should be compatible with the realization data. Kurz (1994) also formulates the rational belief to be the difference between the subjective expectation and the stationary empirical distribution, which provides the empirical foundation of extracting heterogeneous beliefs in this paper.

The heterogeneous beliefs literature has been confined to theoretical studies because it is empirically challenging to measure heterogeneous beliefs among different agents. This paper first tackles the empirical measurement of heterogeneous beliefs as the differences between forecast data and empirical prediction from an econometrics model, taking into account a large macro information set borrowed from Stock and Watson (2002) and updated to 2003. The forecast data are professional forecasts from Blue Chip Economic Indicators (BLUE), Blue Chip Financial Forecasts (BLUF), and Survey of Professional Forecasters (SPF). The paper uses the mean and the interdecile range of the belief indices to characterize heterogeneous beliefs, which are called the belief consensus index (BCI) and the belief dispersion index (BDI), respectively.

Equipped with BCIs and BDIs, this paper conducts several empirical tests on the term structure based on heterogeneous beliefs. It shows that an increase in the BCI of inflation significantly steepens the yield curve, and both the BCI and the BDI of interest rates help explain the mean and the stochastic volatility of the term structure, implying that time-varying risk premia may be explained by higher order uncertainty caused by heterogeneous beliefs about the future conditions of the economy.

The remainder of the paper is organized as follows. Section 2 describes a general equilibrium model with a large number of agents holding heterogeneous beliefs about future endowment shocks. Section 3 constructs the BCIs and the BDIs of various macro variables from professional forecast data. Section 4 uses BCIs and BDIs and their underlying macro variables as factors to explain a linear term structure model. Section 5 estimates a non-linear VAR model using BCIs and BDIs to drive the drift and volatility of the term structure. Section 6 concludes.

2 The term structure model with heterogeneous beliefs

2.1 Model

Consider a real economy with N infinitely-lived agents, where N is large but finite so the law of large numbers holds. Each agent i maximizes the following standard CRRA utility function

$$E_0^i \sum_{t=0}^{\infty} \beta^t \frac{(C_t^i)^{1-\sigma}}{1-\sigma}$$

subject to

$$C_t^i + \sum_{h=1}^{T} Q_t^h B_t^{hi} = Y_t + \sum_{h=1}^{T} Q_t^{h-1} B_{t-1}^{hi}, \tag{1}$$
$$Y_t = Y_{t-1} \mathrm{e}^{y_t}$$

where Q_t^h is the price of a zero-coupon bond with maturity h at t, and B_t^{hi} is such bond holding by agent i. All bonds pay 1 at maturity and are in zero net supply.

Y_t is the endowment realized at t with a growth rate of y_t, which is the same for all agents. So the only heterogeneity in this economy is the heterogeneous expectations of the exogenous endowment growth rate. The agents do not know the true process of the growth rate y_t and they make conditional forecasts of the growth rate of endowment in the future when maximizing their utility today.

The first-order conditions after log-linearization around the symmetric perfect foresight steady state values are

$$\sigma \hat{C}_t^i + r_t^h = E_t^i(\sigma \hat{C}_{t+1}^i + r_{t+1}^{h-1}), \tag{2}$$

while $\hat{C}_t^i$ and r_t^h are the log-deviations of C_t^i and Q_t^h.

The market clearing condition is

$$\bar{\hat{C}}_t \equiv \frac{1}{N} \sum_{i=1}^{N} \hat{C}_t^i = \hat{Y}_t. \tag{3}$$

with $\hat{Y}_t \equiv \log(Y_t) - \log(\bar{Y}) = \hat{Y}_{t-1} + y_t$.

2.2 Heterogeneous belief wedge Δ

The heterogeneous agent model presented here deviates from the conventional representative agent model. After averaging the first-order conditions (2) over all agents, there emerges an extra term Δ_t called the *heterogeneous belief wedge*,

$$\Delta_t = \frac{\sigma}{N} \sum_{i=1}^{N} E_t^i (\hat{C}_{t+1}^i - \bar{\hat{C}}_{t+1}). \tag{4}$$

where $\bar{E}$ is the market average expectation, denoted as $1/N \sum_{i=1}^{N} E^i$.

In a representative agent model when all agents consume the same amount, Δ_t is trivially zero. In the case of homogeneous beliefs, E_t^i is the same for all i, Δ_t also vanishes. Only when the agents hold diverse expectations of the future conditions of the economy, will Δ_t be nonzero. So the representative agent model and the homogeneous beliefs model are both special cases of the heterogeneous beliefs model when the heterogeneous belief wedge Δ_t is zero.

Since consumption is endogenously determined by all state variables in equilibrium, the heterogeneous expected difference between the individual consumptions and the market average consumption reflects the different perceptions of future exogenous shocks held by different agents. Thus the heterogeneous belief wedge Δ_t is an aggregate measure of the disagreement about the future of the economy.

With some tedious algebra shown in Appendix A, Δ_t is the difference between the first-order market expectation (the current market expectation) and the second-order market expectation (the current market expectation of the next period market expectation) of the sum of all future one-period real interest rates.

$$\Delta_t = (\bar{E}_t - \bar{E}_t \bar{E}_{t+1}) \sum_{k=1}^{\infty} (-r_{t+k}^1), \qquad \bar{E}_t = \frac{1}{N} \sum_{i=1}^{N} E_t^i. \tag{5}$$

If the market expectation followed the law of iterated expectations, higher-order expectations would be redundant and Δ_t would be zero. However, market expectations do not follow the law of iterated expectations in general.[1]

2.3 The belief processes and the equilibrium term structure

Following Kurz (1994, 1997), I name different interpretations of the same information heterogeneous beliefs. Then expectations and forecasts are interchangeable, but expectations and beliefs are two distinct concepts. Since agents observe structural changes and regime switches retrospectively, they do not believe the world is stationary.[2] Therefore they want to use their own judgements to correct for the predictions generated by econometric models.[3]

[1] See Lucas (1975), Hellwig (2002), Woodford (2002), Allen, Morris and Shin (2006), and Bacchetta and Wincoop (2004).

[2] A non-stationary world has an infinite dimension of states, which makes the convergence of beliefs impossible.

[3] Batchelor and Dua (1991) surveyed Blue Chip forecasters on the methods they use to predict future macroeconomic variables like GDP growth, interest rates, etc, and found that: “The

As each agent knows that other agents are forecasting with their own beliefs just as he does and that all individual beliefs aggregate to the market belief, the market belief is an endogenous uncertainty to him. Besides forecasting exogenous macroeconomic uncertainties, he forecasts future endogenous market beliefs as well.

In the spirit of Jin (2004), the empirical law of motion for an individual belief, s_t^i, is characterized as follows,

$$s_{t+1}^i = \lambda_c s_t^i + \eta_{t+1}^z + \eta_{t+1}^i,$$

where $\eta_{t+1}^z \sim iid\mathcal{N}(0, \sigma_{t+1}^2)$ and $\eta_{t+1}^i \sim iid\mathcal{N}(0, \sigma_{i,t+1}^2)$. With the law of large numbers, $\frac{1}{N}\sum_{i=1}^N \eta_t^i$ is set to be 0. Namely, the increment of individual belief s_t^i consists of a common component, η_{t+1}^z, and an idiosyncratic component, η_{t+1}^i. The common component captures the correlation of beliefs across agents as they communicate with one another and can be influenced by opinions held by others.

The market belief z_t by definition is the average of individual beliefs s_t^i, then

$$z_{t+1} = \frac{1}{N}\sum_{i=1}^N(\lambda_c s_t^i + \eta_{t+1}^z + \eta_{t+1}^i) = \lambda_c \frac{1}{N}\sum_{i=1}^N s_t^i + \eta_{t+1}^z = \lambda_c z_t + \eta_{t+1}^z, \tag{6}$$

The empirical process[4] of the growth rate of the endowment y_t takes a simple AR(1) process,

$$y_{t+1} = \lambda_a y_t + \eta_{t+1}^y, \tag{7}$$

where η_t^y and η_t^z are correlated.

Then each agent forms his expectations of the growth rate y_{t+1}^i and of the market belief z_{t+1}^i based on the empirical predictions in (7) and (6) and his own belief today s_t^i. This is formulated as the perception model of agent i as follows,

$$y_{t+1}^i = \lambda_a y_t + \lambda_y s_t^i + \rho_{t+1}^{yi}, \tag{8a}$$

$$z_{t+1}^i = \lambda_c z_t + \lambda_z s_t^i + \rho_{t+1}^{zi}, \tag{8b}$$

where ρ_t^{yi} and ρ_t^{zi} are correlated. y_{t+1}^i is how agent i perceives the growth rate y in date $t+1$ and z_{t+1}^i is how the agent thinks the future dynamics of the market

single most important contribution to forecasting technique was Judgement (51%), followed by Econometric Modelling (28%) and Time Series Analysis (21%)." Rudin (1992) found that simple time-series models are inconsistent with SPF forecasts, and there is a great deal of diversity in forecasters' beliefs.

[4] The empirical process is defined as follows. Using moments which are computed empirically from the data one deduces empirically all the finite dimensional distributions of state variables. Given these empirical distributions one extends them to a probability measure over infinite sequences. It can be shown that this probability measure is stationary. In the case of first order Markov process the estimation procedure is reduced to using all the historical data to estimate a single transition function and from it deduce the implied invariant distribution. When the economy undergoes structural changes such stationary transition function is actually the average transition over different structures realized in the past. The outlined procedure is applicable to all processes which do not diverge, such as growth rates.

belief will evolve. Denote by m the stationary probability implied by the empirical process hence $E^m(y^i_{t+1}|y_t) = \lambda_a y_t$ and $E^m(z^i_{t+1}|z_t) = \lambda_c z_t$. It then follows from (8) that the agent's belief s^i_t satisfies

$$E^i(y^i_{t+1}|y_t, s^i_t) - E^m(y^i_{t+1}|y_t) = \lambda_y s^i_t,$$
$$E^i(z^i_{t+1}|z_t, s^i_t) - E^m(z^i_{t+1}|z_t) = \lambda_z s^i_t.$$

This reveals that the belief s^i_t expresses how the agent's forecasts deviate from the stationary forecasts of the two state variables of the model: the growth rate and the market belief. We also see that we can approximately measure the belief of an agent about the future of any state variable by taking the difference between the agent's forecast of that variable and the stationary forecast. This fact is central to the construction of the belief indices in this paper hence to the empirical results reported later.

Observing retrospectively the non-stationary nature of the world, an agent would like to use his belief s^i_t as well as past state variables, growth rate y and market belief z in this model, to predict future state variables in a heterogeneous belief world. Even when his belief s^i_t equals the market belief z_t, he will consider it a coincidence and continue using his belief to predict the future. Therefore the crucial conditions for the existence of heterogeneous beliefs are $\lambda_y \neq 0$ and $\lambda_z \neq 0$ instead of $s^i_t = z_t$.

It is shown in Appendix C that

Proposition 1 *The heterogeneous belief wedge Δ_t is a linear function of z_t,*

$$\Delta_t = b z_t, \quad b = \frac{\sigma\lambda_y\lambda_z[\lambda_a(1-\lambda_c-\lambda_z)+\lambda_z]}{(1-\lambda_a)(1-\lambda_c-\lambda_z)(1-\lambda_c+\lambda_z)}. \tag{9}$$

Clearly, $\lambda_y \neq 0$ and $\lambda_z \neq 0$ are necessary conditions for $\Delta_t \neq 0$. $\lambda_y \neq 0$ requires agents to use their own beliefs to predict the future growth rate of the economy. $\lambda_z \neq 0$ means that agents must have diverse beliefs about future market beliefs.

Using recursive induction I derive in Appendix D the following expression for the equilibrium term structure in terms of total returns,

$$r^h_t = N^h y_t + M^h z_t + H^h \Delta_t \tag{10}$$

$$\text{with} \quad N^h = \sigma\lambda_a \frac{1-(\lambda_a)^h}{1-\lambda_a},$$
$$M^h = \frac{\sigma\lambda_y}{1-\lambda_a}\left(\frac{1-(\lambda_c+\lambda_z)^h}{1-\lambda_c-\lambda_z} - \frac{\lambda_a\big((\lambda_a)^h-(\lambda_c+\lambda_z)^h\big)}{\lambda_a-\lambda_c-\lambda_z}\right),$$
$$H^h = \frac{1-(\lambda_c+\lambda_z)^h}{1-\lambda_c-\lambda_z}.$$

The equilibrium interest rate has three components – a rational expectation component from the exogenous state variable y_t, a market misperception component of z_t, and a heterogeneous beliefs wedge component of Δ_t. Under rational expectations when $\lambda_y = 0$ and $\lambda_z = 0$, M^h and Δ_t are both zero. Thus the equilibrium interest rates only depend on the exogenous macro state variable. However under heterogeneous beliefs when both λ_y and λ_z are nonzero, interest rates are functions of heterogeneous beliefs about the future macro state variable.

Independent of the model specifications of the processes of macro state variables and beliefs, with heterogeneous beliefs the h-period interest rate is

$$r_t^h = r_t^1 + \bar{E}_t r_{t+1}^1 + \bar{E}_t \bar{E}_{t+1} r_{t+2}^1 + \cdots + \bar{E}_t \bar{E}_{t+1} \cdots \bar{E}_{t+h-2} r_{t+h-1}^1. \quad (11)$$

This is the heterogeneous Expectations Hypothesis. Since the law of iterated expectations does not apply to market expectations, the existing empirical tests of the Expectations Hypothesis are misspecified. The correct model should incorporate all future Market expectations up to the maturity of the long-term interest rate.

3 Construction of belief indices

As beliefs data are not directly available, existing research on heterogeneity from either heterogeneous beliefs or private information lacks empirical studies to support the theory. This section makes the first attempt to construct belief indices from individual survey forecasts so that empirical tests of various model implications of heterogeneous beliefs are possible.

3.1 Forecast data description

Three professional forecast data sets are analyzed throughout the paper, the BLUF, the BLUE, and SPF.[5] The forecasts were made by economists in major corporations, financial institutions and economic consultancies; so these professional forecasts presumably display a high level of rationality and a better understanding of the market.

The analysis is focused on the interest rates (the federal funds rate, 3-month treasury bill rate, and AAA bond rate[6]), output (real GDP growth rate, industrial production index (IPI), and unemployment rate), and inflation (GDP deflator and CPI), which may well summarize the professional expectations about the status of the aggregate economy.

Two assumptions are made. First, all market participants know the empirical predictions of the economy, which are nonjudgemental econometric forecasts given a stationary economic environment. Second, they treat such empirical predictions as the benchmark, and then add their own judgements to it. As mentioned in the theory section, Batchelor and Dua (1991) and Rudin (1992) both documented that in practice the majority of professional forecasters use judgements together with econometric models to predict future macroeconomic variables.

[5] SPF contains quarterly forecasts only. Thus it is used when monthly forecasts are unavailable.

[6] The forecasts of all longer term treasury rates, from 1-year to 10-year, start as late as Jan 1988. Since they are 5 years shorter than other forecasts, I choose to use the forecasts of AAA bond rates for long-term rate forecasts in my analysis.

3.2 Construction of the individual belief index

I use $E_t^{Xhi} = E(X_{t+h}|I_t, s_t^{Xhi})$ to denote the forecast for variable X, which may be real GDP growth or federal funds rate, h periods ahead of time t given the information set I_t and the agent's belief about h-period ahead X at time t, s_t^{Xhi}. I use $E_t^{Xhm} = E(X_{t+h}|I_t)$ for the econometric prediction of variable X, h periods ahead of time t given information set I_t only. The difference between the econometric prediction E_t^{Xhm} and the forecast E_t^{Xhi} is the pure effect of the belief, called the individual belief index (BI). Mathematically

$$E_t^{Xhi} - E_t^{Xhm} = f(s_t^{Xhi}) \equiv \mathrm{BI}_t^{Xhi}. \tag{12}$$

BI_t^{Xhi} is assumed to be increasing in s_t^{Xhi}, and in general changes with time, forecasters, macro variables, and forecast horizons. However we should expect the belief indices to be correlated across time, forecasters, forecast horizons, and various macroeconomic variables.

Notice that the BI is different from forecast errors. BI is the difference between the forecast and the econometric prediction, while the forecast error is the difference between the forecast and the realization. The econometric prediction and the realization are different from each other most of the time. Rational forecasters, even though utilizing all available information, make forecasting errors. Take real GDP growth as an example. A positive belief about real GDP growth means the forecaster is optimistic about real economic growth in the future. This does not mean that the real growth is expected to be higher next period than what is realized today. Instead, it means the real GDP growth is expected to be higher than what is predicted by the empirical forecast model with no judgements. Or, the forecaster expects the real GDP growth to be higher than normal. In terms of the belief about the federal funds rate, it is natural to interpret it as the belief about future monetary policy shock. A positive belief about the federal funds rate indicates that the forecaster expects a more contractionary monetary policy than what can be deduced from the econometric prediction. Therefore, the interpretation of belief indices about different macroeconomic variables is quite different even though they share some common components. The BI is constructed in three steps: forming the information set, estimating the stationary econometrics prediction, and extracting the index.

3.2.1 The information set of macro variables

The first step is to find the information set, I_t, which is expected to include all public information, current and past, that agents can possibly access. I use the data set in Stock and Watson (2002) and update it to 2003. Following Stock and Watson (2002), in my updated information set all time series are subjected to possible transformation in order to obtain stationary series and are then screened for outliers.[7] As a result, all real series are transformed to percentage changes over a month through the first difference of logarithms, all price series are taken to be the second difference of logarithms. To obtain a balanced panel, I chose 139 times

[7] I replaced the observations exceeding six times the interquartile range from the median by the median of the series.

series from January 1964 to April 2003 according to the availability of the data. A detailed description of the difference between the Stock and Watson balanced data set and my information set is in Appendix F.

These 139 time series form a balanced panel representing 12 main categories of macroeconomic time series: Consumption, Employment, Exchange Rates, Hourly Earnings, Housing Starts, Interest Rates, Inventory, Money Aggregates, Orders, Prices, Real Output, and Stock Prices. Stacking them, I obtain a large information matrix of the dimension 472 by 139. A principal component analysis (PCA) is carried out to reduce the rank of the matrix and to keep as much information as possible.

The four greatest factors extracted from PCA are able to explain 36.5% of the variation in the information matrix. The marginal contribution of the fifth greatest factor is less than 5%, meaning that not much explanatory power is gained from having more than four factors. Therefore, I pick the four greatest factors as the information set for the sake of efficiency. Coincidentally Stock and Watson (2002) also picked four factors for their forecasting information set. As in Stock and Watson (2002), I use both factors and lagged macro variables to predict future macro variables.

3.2.2 Stationary Prediction

I use a least-square regression model to estimate the empirical prediction E_t^{Xhm} about macro variable X, h periods in the future based on the information set I_t. Recall that BLUF and BLUE are monthly forecasts of quarterly averages, which creates the problem of changing forecast horizons. For example, with the notations introduced in the previous section, $E^{X,1Q,i}_{\text{Jan},1990}$, $E^{X,1Q,i}_{\text{Feb},1990}$ and $E^{X,1Q,i}_{\text{Mar},1990}$ are forecasts of the quarterly average X of the second quarter in 1990, but made at the end of January, February and March 1990 respectively. To match the forecast data and accommodate the changing forecast horizons, the empirical predictions use the following projections:

$$\begin{aligned}
& E_t^{X,1Q,m} = E(\bar{Y}_{t+3}|I_t) \quad E_t^{X,4Q,m} = E(\bar{Y}_{t+12}|I_t) \\
\text{where} \quad & \bar{Y}_t = (X_t + X_{t+1} + X_{t+2})/3 \quad t = \text{Jan, Apr, Jul, Oct} \\
& \bar{Y}_t = (X_{t-1} + X_t + X_{t+1})/3 \quad t = \text{Feb, May, Aug, Nov.} \\
& \bar{Y}_t = (X_{t-2} + X_{t-1} + X_t)/3 \quad t = \text{Mar, Jun, Sep, Dec}
\end{aligned}$$

Notice that $\bar{Y}_t$ is the quarterly average of X that stays the same for all three months within the same quarter. That is, $\bar{Y}_{\text{Jan}} = \bar{Y}_{\text{Feb}} = \bar{Y}_{\text{mar}}$, $\bar{Y}_{\text{Apr}} = \bar{Y}_{\text{May}} = \bar{Y}_{\text{Jun}}$, and so on. Therefore $\bar{Y}_{t+3}$ is the quarterly average of X next quarter and $\bar{Y}_{t+12}$ is the quarterly average of X four quarters in the future.

The general information set is the follows,[8]

$$\begin{aligned}
& I_t = \{F1_t, F2_t, F3_t, F4_t, X_t, \bar{Y}_{t-3}, \bar{Y}_{t-6}, \bar{Y}_{t-9}\} \quad t = \text{Jan, Apr, Jul, Oct} \\
& I_t = \{F1_t, F2_t, F3_t, F4_t, \frac{X_t + X_{t-1}}{2}, \bar{Y}_{t-3}, \bar{Y}_{t-6}, \bar{Y}_{t-9}\} \\
& \qquad t = \text{Feb, May, Aug, Nov} \\
& I_t = \{F1_t, F2_t, F3_t, F4_t, \bar{Y}_t, \bar{Y}_{t-3}, \bar{Y}_{t-6}, \bar{Y}_{t-9}\} \quad t = \text{Mar, Jun, Sep, Dec.}
\end{aligned}$$

[8] For quarterly series such as GDP growth and deflator, X_t is not used in I_t.

Each I_t consists of four current factors from the information matrix and three quarterly lags of $\bar{Y}$ and the realized X_t in current quarter for monthly series. The bayesian information criterion (BIC) is used to determine the number of factors and the number of lagged variables needed for an efficient information set. As the empirical prediction is a nonjudgemental prediction assuming a stationary environment, a real-time prediction that permits time dependent models is inappropriate for this purpose.[9]

The difference between the individual forecast and the stationary prediction is the individual belief index, BI_t^{Xhi}. Recall from the theory section that $\mathrm{BI}_t^i = s_t^i$ with the normalization of $\lambda_y = 1$. The constructed BI is the same as the belief introduced in the theoretical model. Notice that the problem of varying forecast horizons in forecast data and empirical predictions is carried over to belief indices. Thus the belief indices may contain the monthly effect within a quarter.

3.3 Belief consensus index

The BCI_t^{Xh} is defined as the average of the individual belief indices. That is,

$$\begin{aligned} BCI_t^{Xh} &= \frac{1}{N}\sum_{i=1}^{N} BI_t^{Xhi} = \frac{1}{N}\sum_{i=1}^{N}(E_t^{Xhi} - E_t^{Xhm}) \\ &= \frac{1}{N}\sum_{i=1}^{N} E_t^{Xhi} - E_t^{Xhm}. \end{aligned} \tag{13}$$

The first term in (13) is the average forecast among individuals, which is called the consensus forecast in BLUF and BLUE, and the mean forecast in SPF. Therefore the BCI is the difference between the mean forecast and the empirical prediction that is common to all. With the same normalization as the individual beliefs, $\lambda_y = 1$, BCI_t is in fact z_t in the theoretical model.

Table 1 presents some summary statistics of all BCIs next quarter. The sample period is January 1983 – April 2003.[10] The BCIs do not display significant monthly effects when regressed on monthly dummies within a quarter. That means the monthly effects from the changing forecast window are insignificant in the mean beliefs to affect any results here.

For variables that are persistent over time, such as the interest rates and the unemployment rate, the time averages of their BCIs next quarter are close to zero and the variances are small. For more volatile variables like real GDP growth, industrial production and inflation, their BCIs next quarter are substantially different from zero on average and display much greater volatilities over time. The difference in the time averages of BCIs may suggest that forecasters use more

[9] Kurz and Motolese (2004) constructed two types of BCIs from one-shot empirical prediction and real-time empirical prediction respectively. They found that the effects of one-shot BCI and real-time BCI on the federal funds future's market and the 3-month treasury bill market are intrinsically the same, with the real-time BCI performing slightly better in terms of the explanation and prediction of the risk premia of the asset markets.

[10] BLUF forecasts started in January 1983 and the information matrix is available up to April 2003.

Table 1 Summary statistics of belief consensus indices (BCIS), next quarter

		Time Ave	Std Dev	Corr
Interest rates	Fed funds rate	0.01	0.51	0.63
	treasury bill 3 Mon (TB3M)	0.04	0.34	0.58
	AAA	0.07	0.20	0.65
Output	Real GDP (RGDP)	−1.02	1.11	0.25
	IPI	−0.53	2.03	0.12
	Unemp	0.07	0.21	0.50
Inflation	GDP deflator (DEF)	0.46	0.61	0.76
	CPI	0.32	1.02	0.84

The sample period is January 1983 – April 2003. The BCIs of industrial production index (IPI) and unemployment rate (Unemp) are extracted from BLUE consensus forecasts only. Other BCIs are extracted from BLUF and BLUE consensus forecasts combined. Corr stands for the first-order monthly serial correlation

Table 2 Correlation Matrix of (BCIS), next quarter

	Fed Funds	TB3M	AAA	RGDP	IPI	Unemp	Def
Fed Funds	1						
TB3M	**0.82**	1					
AAA	0.38	0.27	1				
RGDP	0.15	0.24	0.07	1			
IPI	0.07	0.18	0.04	**0.88**	1		
Unemp.	–0.49	–0.43	–0.04	–0.45	–0.46	1	
Def.	0.32	0.06	0.39	–0.21	–0.11	–0.10	1
CPI	0.20	0.17	0.42	0.01	0.14	–0.19	**0.60**

Correlation coefficients greater than 0.50 in absolute value are in bold

judgements to predict more volatile time series than the more persistent ones, while the difference in the standard deviations of BCIs hints that such judgements vary more for the more volatile series than for the less volatile ones.

Except for the output BCIs, all BCIs are highly persistent over time and show monthly serial correlations of 0.58 or higher. This observation implies that the judgements about the future conditions of the economy are slow moving processes. Once a forecaster forms an opinion about the future, he tends to stick to it for a relatively long time.

Table 2 reports the correlation matrix of the BCIs where the categorical effect is manifest. The BCIs of short-term interest rates (federal funds rate and 3-month treasury bill rate), the BCIs of output variables (real GDP growth, industrial production, and unemployment), and the BCIs of inflation rates (GDP deflator and CPI) are highly correlated within each category. It is clearly rational to have similar judgments on variables that are alike. Moreover, the BCI of unemployment rate is also highly correlated with the BCIs of federal funds rate and three-month treasury bill rate with correlation coefficients of −0.49 and −0.43 respectively.

To give us some idea of the quality of the BCIs extracted, Table 3 shows the correlation coefficients of the BCIs and the realizations of their underlying variables. The correlation in most cases is weak. For example, the correlation coefficient of the unemployment rate and its BCI next quarter is −0.04, and the correlation coefficient of CPI and its BCI is −0.09. The negative correlation between short-

Table 3 Correlation of the BCIs and the realizations, next quarter

Fed Funds	TB3M	AAA	RGDP	IPI	Unemp	Def	CPI
−0.18	−0.28	0.20	−0.20	−0.18	−0.04	0.23	−0.09

Real GDP growth realization and GDP deflator realization are quarterly data while all other variables are monthly data

term interest rates and their belief indices indicates a perceived faster speed of mean reversion of short rates.

3.4 Belief dispersion indices

The BDI_t^{Xh} is defined as the difference between a 90 percentile belief and a 10 percentile belief about X in h periods. It is a more robust measure for disagreement of beliefs than the cross-sectional variance because this interdecile range eliminates the upper 10% and lower 10% outliers automatically. Mathematically,

$$\begin{aligned}\text{BDI}_t^{Xh} &= BI_t^{Xh,90\%} - BI_t^{Xh,10\%} \\ &= (E_t^{Xh,90\%} - E_t^{Xhm}) - (E_t^{Xh,10\%} - E_t^{Xhm}) \\ &= E_t^{Xh,90\%} - E_t^{Xh,10\%}. \end{aligned} \tag{14}$$

It shows that the interdecile range of individual beliefs is the same as the interdecile range of individual forecasts because the empirical prediction is common to all forecasters.

The BDIs of interest rates next quarter, unlike the BCIs, do show significant monthly effects when regressed on monthly dummies – the dispersion decreases from the first month of the quarter to the third month of the quarter. This phenomenon is expected as more and more information becomes available, agents' beliefs about interest rates converge. To correct for this, I choose the first month as the base for BDI, then add to it significant coefficients of the dummies for the second month and the third month. This guarantees the adjusted BDIs to be positive across time. The BDIs of macro variables have no significant monthly effect, therefore do not need to be adjusted. Since monthly panel forecasts about industrial production and unemployment rate are not available, the BDIs of industrial production and the unemployment rate cannot be constructed.

Figure 1 shows the BDIs next quarter after seasonal adjustment. The BDIs of short-term interest rate – Fed funds rate and 3-month treasury bill rate – are very similar, thus only federal funds BDIs are plotted for short rate BDIs. As heterogeneous forecasts are well documented in some survey studies.[11] It is not surprising to find substantial time-varying dispersions of beliefs about interest rates and macro variables in these plots. The 1980s is a more volatile period than the 1990s, higher BDIs in 1980s reflect more perceived uncertainty about future conditions of the economy in 1980s as Swanson (2004) suggested.

[11] Zarnowitz and Phillip (1992) found in SPF that "the dispersion among the individual predictions is often large and it typically increases with forecast horizon,..." Mankiw, Reis and Wolfers (2003) analyzed inflation rates from several data sources and reported " substantial disagreement among both consumers and professional economists about expected future inflation".

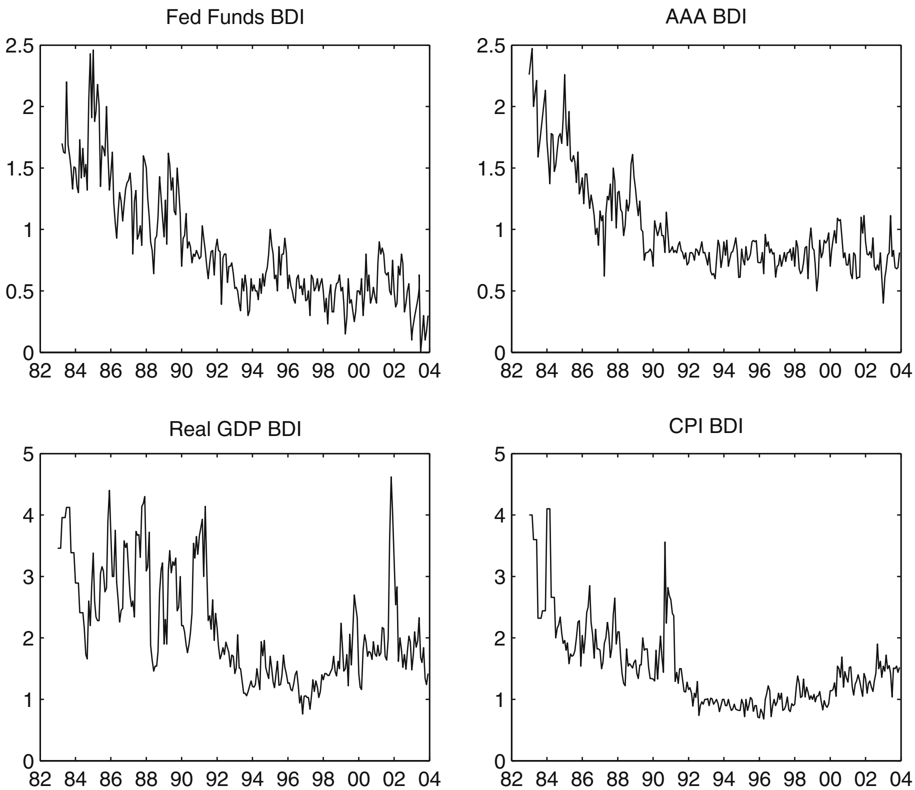

Fig. 1 Belief dispersion indices, next quarter. The sample period is January 1983 – Apr 2003. The forecast data used to extract the BDIs of real GDP growth rate and CPI during Jan 1983 – June 1984, and the BDI of AAA bond rate during Jan 1983 – Dec 1983 are from SPF. Since SPF forecasts are made quarterly instead of monthly, the monthly forecasts during these periods are treated as duplicates of quarterly forecasts within the quarter. All other forecast data are from BLUF

A general regression model is proposed as follows,

$$\mathrm{BDI}_t^X = \alpha_0 + \alpha_1 dC_{t-1} + \alpha_2 dT_{t-1} + \alpha_3 dR_{t-1} + \alpha_4 X_{t-1}, \tag{15}$$

where

dC: Fed chairman dummy. 0: Volcker, 1: Greenspan

dT: Transparent monetary policy dummy.
0: 1983/04-1994/01, 1: 1994/02-2003/04

dR: NBER recession dummy. 0: no recession, 1: recession.

Since real GDP growth rate is released quarterly, a constructed monthly output variable is used to replace real GDP growth rate on the right hand side of the relevant regression. The output variable is a principal component of year-over-year growth rate of IPI and unemployment rate.

Table 4 Belief Dispersion Indices on Dummies, Next Quarter

BDIs	dChair	dTrans	dRecess	X_{t-1}	$\bar{R}^2$
Fed Funds	–0.45*	–0.16*	0.05	0.07*	0.77
	(0.04)	(0.03)	(0.04)	(0.01)	
TB3M	–0.34*	–0.11*	0.04	0.07*	0.70
	(0.04)	(0.03)	(0.05)	(0.01)	
AAA	–0.31*	0.12*	–0.01	0.13*	0.70
	(0.05)	(0.03)	(0.06)	(0.01)	
RGDP	–0.80*	–0.57*	1.06*	–0.06*	0.56
	(0.12)	(0.10)	(0.18)	(0.01)	
CPI	–1.02*	–0.03	0.45*	0.20*	0.54
	(0.11)	(0.07)	(0.14)	(0.03)	

The sample period is April 1983 – December 2003. Newey-West HAC standard errors are reported in parentheses. * represents 1% significance level

Table 4 shows that almost all BDIs are highly correlated with past monetary regimes and their underlying variables. Higher levels of interest rates and CPI result in more dispersed opinions on these variables in the future. Unanimously, the Greenspan era decreases the dispersion of beliefs about all variables. A more transparent monetary policy leads to significantly lower BDIs except the BDI of AAA next quarter. The NBER recession dummy has a strong significant effect on enlarging the disagreement about inflation and real output, but has no significant effect on the BDIs of interest rates.

4 Linear term structure dynamics

The heterogeneous term structure model has shown in (10) that the dynamics of the term structure are determined by macro state variables and market beliefs of these state variables. Following Ang and Piazzesi (2003) I adopt real output and inflation as the macro state variables in my empirical model. However, instead of using latent yield factors as in Ang and Piazzesi (2003), I use the belief indices – BCIs and BDIs – of real output and inflation to explain the term structure dynamics.

4.1 Data

The model uses yield data of maturities 1-month, 3-month, 1-year, 3-year, and 5-year taken from the Fama CRSP treasury bill and zero coupon files. The two monthly macro state variables are inflation (INF) and real output (OUT). INF is the year-over-year percentage change of the seasonally adjusted total CPI. OUT is defined as the first principal component of the unemployment rate and the year-over-year percentage change of the seasonally adjusted total IPI, which explains 91% of the total variance.

The BCIs of inflation ($\mathrm{BCI}^{\mathrm{INF}}$) and real output growth ($\mathrm{BCI}^{\mathrm{OUT}}$) next quarter are used as the proxies for the means of market beliefs of macro state variables.

$\mathrm{BCI}^{\mathrm{INF}}$ is the first factor of BCIs of CPI and the GDP deflator, explaining 86% of the total variance, and $\mathrm{BCI}^{\mathrm{OUT}}$ is the first factor of the BCIs of real GDP growth, of industrial production growth, and of the unemployment rate, accounting for 90% of the total variance. The BDIs of real GDP growth ($\mathrm{BDI}^{\mathrm{OUT}}$) and of CPI ($\mathrm{BDI}^{\mathrm{INF}}$) are used as measures of the dispersion of market beliefs of macro economy.

4.2 Model

Following the theoretical model, I propose two empirical models. One uses macro state variables and the BCIs of macro state variables only, and the other uses both BCIs and BDIs together with macro state variables.

$$
\begin{aligned}
R_t^h &= C^h + A^h \mathrm{BCI}_t + B^h Y_t + \Sigma_t^h \\
R_t^h &= C^h + A^h \mathrm{BCI}_t + B^h \mathrm{BDI}_t + D^h Y_t + \Sigma_t^h
\end{aligned} \tag{16}
$$
$$
\text{with}\quad \mathrm{BCI}_t = \begin{bmatrix} \mathrm{BCI}_t^{INF} \\ \mathrm{BCI}_t^{\mathrm{OUT}} \end{bmatrix} \quad \mathrm{BDI}_t = \begin{bmatrix} \mathrm{BDI}_t^{\mathrm{INF}} \\ \mathrm{BDI}_t^{\mathrm{OUT}} \end{bmatrix} \quad Y_t = \begin{bmatrix} \mathrm{INF}_t \\ \mathrm{OUT}_t \end{bmatrix}.
$$

R_t^h is a vector of five annual continuous compounding yields with maturities from 1-month to 5-year.[12]

Recall that in the theory section I apply log-linearization to derive a close-form solution of the heterogeneous term structure model, as most macro economists do. As a result, second and higher moment effects are ignored. However, theoretically the entire distribution of heterogeneous beliefs matters. By adding back BDIs to the empirical model, I compensate for the effect of the dispersion of heterogeneous beliefs, the second moment of the distribution, on the term structure. Therefore, I use two moments of heterogeneous beliefs to represent the entire distribution.

Generalized method of moments (GMM) is applied to estimate the system of equations above, using Bartlett kernel and variable Newey-West bandwidth to adjust for the serial correlation. The first model is named the mean belief model while the second the full belief model. The model without any belief indices is denoted the macro model.

4.3 Estimation results

Table 5 reports the estimates of the full belief model and the adjusted R^2 for all three models. Apparently the full belief model with belief dispersion indices best explains the dynamics of the term structure. The average $\bar{R}^2$ of the full belief model is 0.63, much higher than that of the mean belief model 0.49 and that of the macro model 0.40. The overall fitness of the full belief model improves as yield maturity increases. For example, $\bar{R}^2$ is 0.59 for 1-month yield, 0.65 for the 1-year yield, and 0.66 for the 5-year yield. This implies that longer-term yields load more heavily on heterogeneous beliefs of macro state variables.

[12] r_t^h in the theoretical model is the total return continuously compounded minus its steady state value. R_t^h is the annual continuous compounding yield. Thus $r_t^h = hR_t^h - h\bar{R}^h$. Then the difference between r_t^h in the theory and R_t^h is a scale factor h and a constant $\bar{R}^h$.

Table 5 Full belief model estimates

	C	BCI^{INF}	BCI^{OUT}	BDI^{INF}	BDI^{OUT}	INF	OUT
R1m	0.75	0.01	0.23*	0.82*	0.45	0.65*	0.25*
	(0.88)	(0.17)	(0.05)	(0.27)	(0.33)	(0.21)	(0.07)
R3m	0.55	0.03	0.23*	0.72*	0.57***	0.75*	0.28*
	(0.86)	(0.17)	(0.05)	(0.27)	(0.32)	(0.21)	(0.07)
R1y	0.48	0.09	0.22*	0.67**	0.71**	0.78*	0.35*
	(0.83)	(0.17)	(0.05)	(0.30)	(0.29)	(0.21)	(0.06)
R3y	1.00	0.31**	0.16*	0.72*	0.74*	0.73*	0.35*
	(0.80)	(0.16)	(0.05)	(0.26)	(0.26)	(0.19)	(0.06)
R5y	1.31***	0.45*	0.14*	0.74*	0.74*	0.71*	0.33*
	(0.78)	(0.15)	(0.05)	(0.23)	(0.24)	(0.19)	(0.06)

	$\bar{R}_2^{\,2}$	$\bar{R}_1^{\,2}$	$\bar{R}_0^{\,2}$
R1m	0.59	0.46	0.38
R3m	0.62	0.50	0.42
R1y	0.65	0.51	0.45
R3y	0.65	0.49	0.40
R5y	0.66	0.50	0.38

General Method of Moments (GMM) is used to estimate all yield equations in one system
*, **, and *** represent 1%, 5%, and 10% significance levels, respectively. Newey-West heteroskedasticity and autocorrelation consistent (HAC) standard errors are in parentheses. Sample is 1983:01–2003:04. $\bar{R}_0^{\,2}$, $\bar{R}_1^{\,2}$, and $\bar{R}_2^{\,2}$ report the adjusted R^2 of the macro model, the mean belief model, and the full belief model respectively

The mean belief about inflation, BCI^{INF}, affects longer-term yields with a significant positive effect. The parameters of BCI^{INF} are increasing with the maturity of the yields, from 0.01 for the 1-month yield to 0.45 for the 5-year yield. Thus, when the market expects inflation to be higher than normal in the near future, even though other beliefs and the real economy today do not change, long-term interest rates go up and the longer-term rates increase more than the shorter-term rates. In other words, an increase in the mean belief about inflation steepens today's yield curve.

The mean belief about real output, BCI^{OUT}, is significant across all yields with decreasing coefficients. That is, holding other variables constant, an increase in the market belief about real output raises all yields and increases the shorter-term yields more than the longer-term yields. When the market expects real output to be higher than normal in the near future the yield curve shifts up and flattens.

The dispersion of beliefs about inflation, BDI^{INF}, shows a significant curvature effect on the yield curve. The parameter of BDI^{INF} decreases from 0.82 for 1-month yield to 0.67 for 1-year yield, and then increases to 0.74 for 5-year yield. Therefore, when there is an increased perceived uncertainty about future inflation, short-term and long-term yields increase more than the mid-term yield, making the yield curve less concave.

The dispersion of beliefs about output, BDI^{OUT}, has no effect on the 1-month yield, but some positive effect on mid-term and longer-term yields. That is, a higher expected uncertainty about future real output increases yields mostly in the middle and long run. In particular, the longer the maturity, the greater the impact of BDI^{OUT} on the yield.

An increase in realized inflation (INF) raises yields significantly for all maturities. The realized output (OUT) also positively affect the whole yield curve significantly. This result is expected as nominal interest rates are the sums of real interest rates and inflation rates, and real interest rates are positively correlated.

In sum, BCI^{INF} and BDI^{OUT} act like slope factors, BCI^{OUT} resembles a combination of level and slope factors, and BDI^{INF} behaves like a combination of level and curvature factors.

Using BCIs and BDIs in four quarters as explanatory variables in the belief models yields similar patterns as using next quarter beliefs indices, with slightly better fit of the curve. A plausible explanation is that short-term belief indices tend to be more distorted by temporary shocks than long-term belief indices so that short-term belief indices contain more noises than their longer-term counterparts. I also try adding absolute values of the BCIs in the models to explore any asymmetric effect of beliefs on the term structure. However, no significant results are found in any yield equation in either the mean belief model or the full belief model.

5 Evolving term structure with stochastic volatility

Although any non-linear effect of beliefs on the term structure is not present in my theoretical model because of log-linearization, it is reasonable to expect heterogeneous beliefs to affect the term structure in both linear and nonlinear ways. This section explores a more general term structure model with a time-varying slope and stochastic volatility driven by the belief indices.

5.1 Model setup

Following Cogley (2005), I set up a bivariate VAR of the term structure with evolving parameter matrices and stochastic volatility as follows:

$$Z_t = A_{0t-1} + A_{1t-1} Z_{t-1} + V_{t-1}^{1/2} \Sigma_t \quad \Sigma_t \sim iidN(0, I) \tag{17}$$

where $Z_t = [R_t, s_t]$, with R_t being the annualized short-term yield and s_t being the spread between the short yield and the long yield. The constant matrix A_{0t-1} and the slope matrix A_{1t-1} have the following representations:

$$A_{0t} = \begin{pmatrix} a_{10t} \\ a_{20t} \end{pmatrix} \quad A_{1t} = \begin{pmatrix} a_{11t} & a_{12t} \\ a_{21t} & a_{22t} \end{pmatrix}$$

$$a_{1it} = \alpha_{1i} + \beta_{1i}\text{BCI}_t + \gamma_{1i}\text{BDI}_t$$

$$a_{2it} = \alpha_{2i} + \beta_{2i}\text{BCI}_t + \gamma_{2i}\text{BDI}_t \quad i = 0, 1, 2, 3, 4.$$

Following Cogley and Sargent (2001), the innovation variance V_t is factored as[13]

[13] All positive definite matrices can be decomposed uniquely as BAB' with A being positive diagonal and B being lower triangular with 1's on diagonal.

$$V_t = \begin{pmatrix} 1 & 0 \\ b & 1 \end{pmatrix}^{-1} \begin{pmatrix} h_{1t} & 0 \\ 0 & h_{2t} \end{pmatrix} \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}^{-1},$$

h_{1t} and h_{2t} are assumed to be changing with beliefs as well:

$$h_{1t} = \phi_0 + \phi_1 \mathrm{BCI}_t^2 + \phi_2 \mathrm{BDI}_t^2,$$
$$h_{2t} = \varphi_0 + \varphi_1 \mathrm{BCI}_t^2 + \phi_2 \mathrm{BDI}_t^2.$$

As both BCI and BDI measure the perceived uncertainty in the future, they should explain the volatility of the term structure. This variance specification is attractive because I ensure that V_t is positive definite and allow for time-varying correlation between the VAR innovations.

Dai, Singleton and Yang (2003) find that models with regime shifts of volatility better explain excess returns on bonds than single regime models. In my paper, when heterogeneous beliefs are introduced to the term structure model, they act as extra comprehensive state variables that incorporate the perceived structural changes, including regime switches. Hence the term structure model with heterogeneous beliefs is consistent in spirit with regime switch term structure models.[14]

5.2 Data

Three-month treasury bill rate is used for the short rate R_t and the difference between the 5-year zero coupon rate and the 3-month treasury bill rate is used for the spread s_t. Both yields are continuously compounded from Fama CRSP files.[15]

The model uses the BCI and the BDIs of interest rates next quarter. The interest rate belief indices are the first principal components of the belief indices of the 3-month treasury bill rate and AAA bond rate. It would be ideal to use belief indices that exactly match the realization of interest rates used in the model. However, forecasts of the 5-year treasury note rate start as late as in January 1988, as a result the belief indices constructed from this forecast series also start in January 1988, thus they do not provide as many observations as we want to carry out a more reliable estimation of the term structure.

The nonlinear model only uses beliefs of interest rates rather than all belief indices in previous section because of the high dimensionality of the problem. The proposed nonlinear model has altogether 25 parameters to estimate given one set of heterogeneous belief indices, that is one BCI and one BDI. The number of parameters goes up to 41 for two sets of belief indices, and 57 for three sets. Owing to the curse of dimensionality, it is nearly impossible to obtain stable estimates of a model with more than one set of belief indices.

5.3 Estimation procedure

The model is estimated using the maximum likelihood method in three steps. In the first step, I apply the Seemingly Unrelated Regression to estimate the parameters

[14] However, heterogeneous belief models suggest that it is more of agents' perceived structural changes that should change term structure dynamics. Hence, we expect fluctuations in the term structure to reflect perceived structural changes rather than the realized structural changes.

[15] The 5-year rate is not necessarily long enough to represent long-term, but it is the longest available zero coupon rate that I could obtain.

in A_0 and A_1 assuming constant volatility. Then I compute the variance-covariance matrix of the residuals and factor out the constant innovation variance V.

In the second step, the results from the first step are used as the initial values to run maximum likelihood estimations. I start with a benchmark model assuming rational expectations. That is, the model has a time invariant long-run mean and constant volatility. To estimate this model, I restrict all parameters associated with BCI and BDI to be zero. So agents are restricted to agreeing with each other and are assumed to know the equilibrium map from states to prices at all times. Then I loosen the restrictions on parameters of BCI to estimate a homogeneous beliefs model. In this case, everybody uses the common belief to forecast the future term structure, while this belief deviates from the true measure of the term structure now and then. The most general model is the heterogeneous beliefs model that puts no restrictions on parameters. It has a time-varying mean and slope, and stochastic volatility. Since all three models are nested, I conduct likelihood ratio tests to pick the best model out of the three.

The third step is a refinement step. I look for the best heterogeneous model by exhausting all model specifications of heterogeneous beliefs.

To gain some robustness of the estimates and thus the robustness of the log likelihood function values, many different initial values for the parameters, and various maximization options are tried. In general, the parameter estimates converge with a tolerance of 10^{-13} and the function values converge with a tolerance of 10^{-8}.

5.4 Estimation results

Among the three models, the likelihood ratio tests show that the homogeneous beliefs model explains the dynamics of the term structure model much better than the rational expectations model with a χ^2 of 50.00, and the heterogeneous beliefs model significantly beats the homogeneous model with a χ^2 of 66.74.

Refinement is needed to better examine the effects of heterogeneous beliefs on the term structure. Different model specifications are tried by adding and removing BCI and BDI in matrices $A0$, $A1$, and V. Likelihood ratio tests are used to select better models. In the end, the ‘best’ model turns out to be a heterogeneous beliefs model with BCI and BDI explaining both matrices $A0$ and V. This ‘best’ model relates to and reinforces the linear term structure model in the previous chapter. They both show that the mean and the dispersion of beliefs help explain the term structure linearly. Moreover, the ‘best’ model illuminates that both the mean and the dispersion of beliefs explain the stochastic volatility of the term structure, implying that the endogenous uncertainty generated by people’s expectations of others’ expectations is the source of time-varying risk premia of the term structure, consistent with the excess holding period return results.

Table 6 reports detailed estimates of the ‘best’ heterogeneous beliefs model. Its log likelihood function value is 423.12.

Matrix $A0$ shows the existence of a time-varying mean of the term structure. The BCI and BDI of interest rates significantly affect both the short rate and the spread. Since the BDI measures the perceived uncertainty about future interest rates, an increase in BDI should raise the risk premium to compensate for such perceived risk. If a higher BDI has a greater impact on the long term rate than

Table 6 Heterogeneous beliefs model estimates, 1-month horizon

$$A0 = \begin{bmatrix} -0.042 - 0.238\text{BCI} - 0.031\text{BDI} \\ (0.043)\ (0.029)^*\ (0.044) \\ 0.293 - 0.097\text{BCI} + 0.151\text{BDI} \\ (0.067)^*\ (0.039)^{**}\ (0.066)^{**} \end{bmatrix}$$

$$A1 = \begin{bmatrix} 0.994 & 0.089 \\ (0.011)^* & (0.019)^* \\ -0.061 & 0.886 \\ (0.016)^* & (0.025)^* \end{bmatrix}$$

$$V = \begin{bmatrix} 1 & 0 \\ & \\ 0.430 & 1 \\ (0.067)^* & \end{bmatrix}^{-1} diag \begin{bmatrix} -0.016 - 0.008\text{BCI}^2 + 0.051\text{BDI}^2 \\ (0.002)^*\ (0.003)^{**}\ (0.000)^* \\ 0.054 + 0.009\text{BCI}^2 + 0.010\text{BDI}^2 \\ (0.010)^*\ (0.018)\ (0.005)^{**} \end{bmatrix} \begin{bmatrix} 1 & 0.430 \\ & (0.067)^* \\ 0 & 1 \\ & \end{bmatrix}^{-1}.$$

Standard errors are reported in parentheses
* and **, stands for 1 and 5% significance level, respectively

on the short term rate, the spread increases. The BDI of interest rates on average is 1.21%, more than ten times bigger than that of the BCI of 0.09%. Therefore, considering that the coefficients of BCI are less than ten times greater than those of BCI, the BDI displays a much greater effect in explaining the mean of the term structure than the BCI.

Matrix $A1$ shows that the short rate and the spread are highly persistent series with serial correlations of 0.99 and 0.89, respectively. It also shows that the lagged spread is negatively correlated with the current short rate. These are standard observations of term structure data.

Removing the insignificant parameters, I write out the innovation variance matrix V as follows to examine the effect of the BCI and BDI of interest rates on stochastic volatility,

$$V = \begin{bmatrix} -0.016 - 0.008\text{BCI}^2 + 0.051\text{BDI}^2 & 0.007 + 0.003\text{BCI}^2 - 0.022\text{BDI}^2 \\ 0.007 + 0.003\text{BCI}^2 - 0.022\text{BDI}^2 & 0.051 - 0.002\text{BCI}^2 + 0.019\text{BDI}^2 \end{bmatrix}.$$

A 1% increase in the BCI decreases the stochastic volatilities of the short rate and spread by 0.008 and 0.002, respectively. A 1% increase in the BDI increases the stochastic volatility of the short rate and the spread by 0.051 and 0.019, respectively. The BCI has a positive effect while the BDI has a negative effect on the stochastic covariance between the short rate and the spread. Following the same logic in matrix $A0$, the effect from BDI dominates the innovation variance, thus better explains time-varying risk premia in the term structure.

A similar estimation is conducted on the heterogeneous beliefs model with a 1-year forecast horizon. That is,

$$Z_{t+12} = A_{0t} + A_{1t} Z_t + V_t^{1/2} \Sigma_{t+12},$$

with identical specifications of Z,A matrices, V, and Σ, except that the interest rate BCI and BDI in four quarters are used to match the 1-year horizon.

Table 7 reports the estimation results of the refined heterogeneous beliefs model with a horizon of 1 year. Similar to 1-month horizon, (a) the yield BCI and BDI are significant in $A0$ and V matrices but insignificant in $A1$; and (b) the yield

Table 7 Heterogeneous beliefs model estimates, 1-year horizon

$$A0 = \begin{bmatrix} 0.288 - 1.297\text{BCI} + 0.994BDI \\ (0.209)\ (0.097)^{*}\ (0.129)^{*} \\ 1.100 + 0.685\text{BCI} - 0.086\text{BDI} \\ (0.147)^{*}\ (0.045)^{*}\ (0.028)^{*} \end{bmatrix}$$

$$A1 = \begin{bmatrix} 0.472 & 0.360 \\ (0.050)^{*} & (0.084)^{*} \\ 0.027 & 0.057 \\ (0.017) & (0.056) \end{bmatrix}$$

$$V = \begin{bmatrix} 1 & 0 \\ & \\ 0.247 & 1 \\ (0.049)^{*} & \end{bmatrix}^{-1} diag \begin{bmatrix} 0.646 - 0.111\text{BCI}^2 + 0.064\text{BDI}^2 \\ (0.105)^{*}\ (0.032)^{*}\ (0.020)^{*} \\ 0.355 - 0.098\text{BCI}^2 + 0.050\text{BDI}^2 \\ (0.078)^{*}\ (0.000)^{*}\ (0.015)^{*} \end{bmatrix} \begin{bmatrix} 1 & 0.247 \\ & (0.049)^{*} \\ 0 & 1 \\ & \end{bmatrix}^{-1}.$$

Standard errors are reported in parentheses.
* stands for 1% significance level

BDI explains more of time-varying risk premia.[16] That is, regardless of forecast horizon, both the mean and the dispersion of beliefs about interest rates affect the time-varying drift and the stochastic volatility of the term structure and the dispersion always plays a more significant role.

The differences between the two results are illuminating. In matrix $A0$, the effect of the BCI on the spread changes from negative to positive from 1-month horizon to 1-year horizon, while the effect of the BDI on the spread changes from positive to negative from short term to long term. The changes in the signs of the parameters of the belief indices suggest that the law of motion of the term structure is non-stationary even though the BDI contains some information about structural breaks. The impact of heterogeneous beliefs on forecasting the term structure varies with forecast horizon.

Another difference lies in the coefficients in matrix $A1$, where the serial correlations of short rate and spread are much weaker as the forecast horizon moves from 1-month to 1-year. This leaves more room for the beliefs to explain the interest rates movements. As a consequence, both BCI and BDI display stronger effect on the time-variability of the term structure in $A0$ and V as the forecast horizon increases.

6 Conclusion

This paper studies how heterogeneous beliefs impact term structure dynamics. It creates measures of heterogeneous beliefs represented by the mean and the dispersion of belief indices that are extracted from professional forecasts. It finds that (a) a rise in the mean belief about inflation steepens the yield curve, (b) heterogeneous beliefs about interest rates drive the mean and stochastic volatility of the term structure.

The measures of heterogeneous beliefs can be adopted to study issues in other financial markets, such as equity premia in stock markets and pricing anomalies in

[16] The yield BCI and BDI in four quarters are 0.44 and 2.34% on average respectively.

derivative markets. The main idea is that movements of asset prices result mainly from endogenous uncertainty propagated by agents' heterogeneous beliefs about future conditions of the economy and their corresponding actions.

Appendix

A Derivation of Δ_t

As shown in the wedge expression (4),

$$\Delta_t = \frac{\sigma}{N}\sum_{i=1}^{N} E_t^i(\hat{C}_{t+1}^i - \bar{\hat{C}}_{t+1}).$$

Thus,

$$\frac{1}{N}\sum_{i=1}^{N} E_t^i(\sigma\hat{C}_{t+1}^i) = \frac{1}{N}\sum_{i=1}^{N} E_t^i\left(E_{t+1}^i\sum_{k=0}^{\infty}(-r_{t+1+k}^1)\right) = \bar{E}_t\sum_{k=1}^{\infty}(-r_{t+k}^1)$$

$$\frac{1}{N}\sum_{i=1}^{N} E_t^i(\sigma\bar{\hat{C}}_{t+1}) = \bar{E}_t\left(\frac{1}{N}\sum_{i=1}^{N}\sigma\hat{C}_{t+1}^i\right) = \bar{E}_t\left(\frac{1}{N}\sum_{i=1}^{N}E_{t+1}^i\sum_{k=0}^{\infty}\left(-r_{t+1+k}^1\right)\right)$$

$$= \bar{E}_t\bar{E}_{t+1}\sum_{k=1}^{\infty}(-r_{t+k}^1)$$

Therefore, $$\Delta_t = (\bar{E}_t - \bar{E}_t\bar{E}_{t+1})\sum_{k=1}^{\infty}(-r_{t+k}^1)$$

B Proof of Corollary 1

Corollary 1 *Let $\Theta(X_t) = (\bar{E}_t - \bar{E}_t\bar{E}_{t+1})\sum_{k=1}^{\infty} X_{t+k}$ to simplify the mathematical expression of the infinite wedge of X. Then $\Theta(y_t)$, and $\Theta(z_t)$ are linear functions of the market belief z_t with*

$$\Theta(y_t) = -\frac{\lambda_y\lambda_z}{(1-\lambda_a)(1-\lambda_c)}z_t, \quad \Theta(z_t) = -\left(\frac{\lambda_z}{1-\lambda_c}\right)^2 z_t.$$

Proof From (8), the two-period individual expectation of the growth rate can be derived as follows,

$$E_t^i y_{t+2} = E_t^i E_{t+1}^i y_{t+2} = E_t^i(\lambda_a y_{t+1} + \lambda_y s_{t+1}^i) = (\lambda_a)^2 y_t + \lambda_y(\lambda_a + \lambda_c)s_t^i.$$

Carry on the same derivation recursively, k-period individual expectation is

$$E_t^i y_{t+k} = (\lambda_a)^k y_t + \lambda_y\sum_{j=0}^{k-1}(\lambda_a)^{k-1-j}(\lambda_c)^j s_t^i = (\lambda_a)^k y_t + \lambda_y\frac{(\lambda_a)^k - (\lambda_c)^k}{\lambda_a - \lambda_c}s_t^i,$$

given $\lambda_a \neq \lambda_c$. Similarly,

$$E_t^i z_{t+k} = (\lambda_c)^k z_t + k\lambda_z(\lambda_c)^{k-1} s_t^i.$$

Assuming $|\lambda_a| < 1$, $|\lambda_c| < 1$, and $|\lambda_a/\lambda_c| < 1$ to reassure the stability of infinite expectations, I obtain the follows,

$$E_t^i \sum_{k=1}^{\infty} y_{t+k} = \frac{\lambda_a}{1-\lambda_a} y_t + \frac{\lambda_y}{(1-\lambda_a)(1-\lambda_c)} s_t^i,$$

$$E_t^i \sum_{k=1}^{\infty} z_{t+k} = \frac{\lambda_c}{1-\lambda_c} z_t + \frac{\lambda_z}{(1-\lambda_c)^2} s_t^i.$$

Then the market infinite sum of expectations are,

$$\bar{E}_t \sum_{k=1}^{\infty} y_{t+k} = \frac{\lambda_a}{1-\lambda_a} y_t + \frac{\lambda_y}{(1-\lambda_a)(1-\lambda_c)} z_t,$$

$$\bar{E}_t \sum_{k=1}^{\infty} z_{t+k} = \frac{\lambda_c(1-\lambda_c)+\lambda_z}{(1-\lambda_c)^2} z_t.$$

The relevant second-order expectations are,

$$\begin{aligned}
\bar{E}_t \bar{E}_{t+1} \sum_{k=1}^{\infty} y_{t+k} &= \bar{E}_t \left(y_{t+1} + \frac{\lambda_a}{1-\lambda_a} y_{t+1} + \frac{\lambda_y}{(1-\lambda_a)(1-\lambda_c)} z_{t+1} \right) \\
&= \frac{\lambda_a}{1-\lambda_a} y_t + \frac{\lambda_y}{1-\lambda_a} \frac{1+\lambda_z}{1-\lambda_c} z_t, \\
\bar{E}_t \bar{E}_{t+1} \sum_{k=1}^{\infty} z_{t+k} &= \bar{E}_t \left(z_{t+1} + \frac{\lambda_c(1-\lambda_c)+\lambda_z}{(1-\lambda_c)^2} z_{t+1} \right) \\
&= \frac{(1-\lambda_c+\lambda_z)(\lambda_c+\lambda_z)}{(1-\lambda_c)^2} z_t.
\end{aligned}$$

Therefore, the infinite wedges are

$$\Theta(y_t) = -\frac{\lambda_y \lambda_z}{(1-\lambda_a)(1-\lambda_c)} z_t, \quad \Theta(z_t) = -\left(\frac{\lambda_z}{1-\lambda_c}\right)^2 z_t$$

□

C Proof of Proposition 1

Proposition 1

$$\Delta_t = b z_t, \quad b = \frac{\sigma\lambda_y\lambda_z[\lambda_a(1-\lambda_c)+\lambda_z(1-\lambda_a)]}{(1-\lambda_a)(1-\lambda_c-\lambda_z)(1-\lambda_c+\lambda_z)}. \tag{18}$$

Proof As $\Delta_t = \Theta(-r_t^1)$ from (5), from Corollary 1, Δ_t must be linear in z_t. Assume $\Delta_t = bz_t$ with b constant. With maturity 1,

$$r_t^1 = \bar{E}_t(\sigma y_{t+1}) + \Delta_t.$$

Apply Θ operator on both sides of the above equation,

$$\Theta(r_t^1) = \sigma\Theta(\bar{E}_t y_{t+1}) + \Theta(\Delta_t).$$

Then,

$$\begin{aligned}
-\Delta_t &= \sigma\Theta(\bar{E}_t y_{t+1}) + \Theta(\Delta_t) \\
&= \sigma\Theta(\lambda_a y_t + \lambda_y z_t) + \Theta(bz_t) \\
&= -\frac{\sigma\lambda_a\lambda_y\lambda_z}{(1-\lambda_a)(1-\lambda_c)} z_t - \sigma\lambda_y\left(\frac{\lambda_z}{1-\lambda_c}\right)^2 z_t - b\left(\frac{\lambda_z}{1-\lambda_c}\right)^2 z_t \\
&= -\frac{\sigma\lambda_y\lambda_z}{1-\lambda_c}\left[\frac{\lambda_a(1-\lambda_c-\lambda_z)+\lambda_z}{(1-\lambda_a)(1-\lambda_c)}\right] z_t - \left(\frac{\lambda_z}{1-\lambda_c}\right)^2 \Delta_t.
\end{aligned}$$

To reassure that Δ_t is nonzero, I need to impose following conditions:

$$\lambda_y \neq 0, \quad \lambda_z \neq 0, \qquad \lambda_a(1-\lambda_c-\lambda_z)+\lambda_z \neq 0.$$

Collecting Δ_t terms, I get

$$\Delta_t = \frac{\sigma\lambda_y\lambda_z[\lambda_a(1-\lambda_c-\lambda_z)+\lambda_z]}{(1-\lambda_a)(1-\lambda_c-\lambda_z)(1-\lambda_c+\lambda_z)} z_t.$$

D Derivation of r_t^h

The market average Euler equation together with the market clearing condition, $\bar{\hat{C}}_t = \hat{Y}_t$ for all t shows

$$r_t^h = \bar{E}_t(r_{t+1}^{h-1} + \sigma\hat{Y}_{t+1} - \sigma\hat{Y}_t) + \Delta_t = \bar{E}_t(r_{t+1}^{h-1} + \sigma y_{t+1}) + \Delta_t. \tag{19}$$

Therefore $r_t^1 = \sigma\lambda_a y_t + \sigma\lambda_y z_t + bz_t$. Plug it into (19), I obtain

$$\begin{aligned}
r_t^2 &= \bar{E}_t(r_{t+1}^1 + \sigma y_{t+1}) + bz_t \\
&= \sigma\lambda_a(\lambda_a+1)y_t + \sigma\lambda_y(\lambda_a+1+\lambda_c+\lambda_z)z_t + [b(\lambda_c+\lambda_z)+b]z_t.
\end{aligned}$$

Guess that

$$r_t^h = \sigma\lambda_a\sum_{j=0}^{h-1}(\lambda_a)^j y_t + \sigma\lambda_y\sum_{j=0}^{h-1}\Big(\sum_{i=0}^{h-j-1}(\lambda_a)^i(\lambda_c+\lambda_z)^j\Big)z_t + b\sum_{j=0}^{h-1}(\lambda_c+\lambda_z)^j z_t.$$

The guessed expression is true for $h = 1, 2$. Now assume the above is true for maturities up to h, I will show that it is also true for $h + 1$.

$$
\begin{aligned}
r_t^{h+1} &= \bar{E}(\sigma y_{t+1}) + \bar{E}\Bigg(\sigma\lambda_a \sum_{j=0}^{h-1} (\lambda_a)^j y_{t+1} + \sigma\lambda_y \sum_{j=0}^{h-1} \Big(\sum_{i=0}^{h-j-1} (\lambda_a)^i (\lambda_c \\
&\qquad +\lambda_z)^j \Big) z_{t+1} + b\sum_{j=0}^{h-1} (\lambda_c + \lambda_z)^j z_{t+1} \Bigg) + b z_t \\
&= \sigma\lambda_a \sum_{j=0}^{h} (\lambda_a)^j y_t + \sigma\lambda_y \sum_{j=0}^{h} \Big(\sum_{i=0}^{h-j} (\lambda_a)^i (\lambda_c + \lambda_z)^j \Big) z_t + b\sum_{j=0}^{h} (\lambda_c + \lambda_z)^j z_t .
\end{aligned}
$$

Thus I verified the derivation of r_t^h by induction. Furthermore,

$$
\begin{aligned}
r_t^h &= \sigma\lambda_a \frac{1-(\lambda_a)^h}{1-\lambda_a} y_t + \sigma\lambda_y \sum_{j=0}^{h-1} \frac{1-(\lambda_a)^{h-j}}{1-\lambda_a} (\lambda_c + \lambda_z)^j z_t + b\frac{1-(\lambda_c+\lambda_z)^h}{1-\lambda_c-\lambda_z} z_t \\
&= \sigma\lambda_a \frac{1-(\lambda_a)^h}{1-\lambda_a} y_t + \frac{\sigma\lambda_y}{1-\lambda_a} \Bigg(\frac{1-(\lambda_c+\lambda_z)^h}{1-\lambda_c-\lambda_z} - \frac{\lambda_a\big((\lambda_a)^h - (\lambda_c+\lambda_z)^h\big)}{\lambda_a - \lambda_c - \lambda_z} \Bigg) z_t \\
&\quad + b\frac{1-(\lambda_c+\lambda_z)^h}{1-\lambda_c-\lambda_z} z_t .
\end{aligned}
$$

E Data description

The basic information set used to estimate non-judgmental stationary prediction and belief indices is taken from Stock and Watson data set used to construct the Diffusion Indices from their webpage[17] and then updated to 2003.

The Stock and Watson data set was collected from Data Resources Inc (DRI), which was replaced by Global Insight at the end of 2002. It contains 146 balanced monthly time series from January 1959 to December 1998, covering 14 main categories of macroeconomic time series: average hourly earnings (AHE), employment (EMP), exchange rates (EXR), housing starts (HSS), interest rates (INT), inventory (INV), money aggregates(MON), orders (ORD), output (OUT), personal consumption (PCE), price index (PRI), retail sales (RTS), stock prices (SPR), and miscellaneous (OTH).

Except for the range of the data, January 1959 – December 1998 for Stock and Watson data set and January 1964 – April 2003 for the basic information set in this paper, the differences between the two data sets are recorded in the following table treating the Stock and Watson balanced panel as the base:

The number in parentheses with each category is the number of series in that category. Each series is presented with a number and a code. The number is the ordering number of the series same as in the data appendix in Stock and Watson (2002) and the code is DRI/Global Insight code of the series. There is no change of the series in the category of personal consumption (PCE).

[17] http://www.wws.princeton.edu/mwatson/publi.html.

Category	Added & deleted	Memo
AHE (7)	Add: 203 leh, 206 lehtu, 207 lehtt, 208 lehfr, 209 lehs	All start in January 1964
EMP (28)	Add: 40 lpmi, 46 lptu, 51 lw	lw starts in January 1964. lpmi and lptu have gross outliers.
EXR (5)	Add: 136 exruk; Delete: 133 exrger	exruk has gross outliers. exrger stops in December 1998.
HSS (18)	Add: 75 hsbne, 76 hsbmw, 77 hsbsou, 78 hsbwst, 79 hns, 84 hnr, 85 hniv, 87 contc, 88 conpc, 89 conqc, 90 condo9	hsbne, hsbmw, hsbsou, and hsbwst start in January 1960; hns, hnr, and hniv start in January 1963. contc, conpc, and conqc start in January 1964. condo9 is now available from Global Insight after October 1998
INT (17)	Add: 138 fyff, 139 fycp90, 140 fygm3, 141 fygm6, 142 fygt1; Delete: 148 fyfha, 157 sfyfha	fyff, fycp90, fygm3, fygm6, and fygt1 have gross outliers. fyfha and sfyfha stop after May 2000.
INV (1)	Delete: 91 ivmtq, 92 ivmfgq, 93 ivmfdq, 94 ivmfnq, 95 ivwrq, 96 ivrrq, 97 ivsrq, 98 ivsrmq, 99 ivsrwq, 100 ivsrrq	All stop after July 2001.
MON (10)	Add: 171 fclnq, 172 fclbmc, 176 ccinrv	fclnq and fclbmc have gross outliers. ccinrv is now available from Global Insight before January 1980 and after September 1995.
ORD (4)	Delete: 105 mdoq, 107 mo, 108 mowu, 109 mdo, 110 mduwu, 111 mno, 112 mnou, 113 mu, 114 mdu, 115 mnu, 116 mpcon, 117 mpconq	All stop after March 2001.
OUT (19)	Add: 10 ipmd (now ips34), 25 gmpyq; Delete: 8 ipi, 18 ipxmca	ipmd and gmpyq have gross outliers. ipi is unavailable from Global Insight. ipxmca stops after October 2002.
PRI (20)	Add: 180 pwimsa, 181 pwcmsa	Both have gross outliers.
RTS (0)	Delete: 55 msmtq, 56 msmq, 57 msdq, 58 msnq, 59 wtq, 60 wtdq, 61 wtnq, 62 rtq, 63 rtnq	All stop after July 2001.
SPR (5)	Delete: 125 fspcap, 127 fsput	Both stop after December 2001.
OTH (0)	Delete: 215 hhsntn	hhsntn, index of consumer expectation, not relevant
Total (139)	Add 32 series; Delete 39 series	

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